References

[1]Mack, L. M., "Linear Stability Theory and the Problem of Supersonic Boundary-Layer Transition," *AIAA Journal,* Vol. 13, No. 3, 1975, pp. 278–298.

[2]Mack, L. M., "Boundary Layer Linear Stability Theory," AGARD Rept. 709, June 1984.

[3]Potter, J. L., "Review of the Influence of Cooled Walls on Boundary Layer Transition," *AIAA Journal,* Vol. 18, No. 8, 1980, pp. 1010–1012.

[4]Sanator, R. J., DeCarlo, J. P., and Torrillo, D. T., "Hypersonic Boundary Layer Transition Data for a Cold-Wall Slender Cone," *AIAA Journal,* Vol. 3, No. 4, 1965, pp. 758–760.

[5]Stetson, K. F., and Rushton, G. H., "Shock Tunnel Investigation of Boundary Layer Transition at $M = 5.5$," *AIAA Journal,* Vol. 5, No. 5, 1967, pp. 899–906.

[6]Mateer, G. G., "Effects of Wall Cooling and Angle of Attack on Boundary-Layer Transition on Sharp Cones at $M_\infty = 7.4$," NASA TD D-6908, Aug. 1972.

[7]Stetson, K. F., Kimmel, R. L., Thompson, E. R., Donaldson, J. C., and Siler, L. G., "A Comparison of Planar and Conical Boundary Layer Stability and Transition at a Mach Number of 8," AIAA Paper 91-1639, June 1991.

[8]Kendall, J. M., "Wind Tunnel Experiments Relating to Supersonic and Hypersonic Boundary Layer Transition," *AIAA Journal,* Vol. 13, No. 3, 1975, pp. 290–299.

[9]Demetriades, A., "New Experiments on Hypersonic Boundary Layer Stability Including Wall Temperature Effects," *Proceedings of the Heat Transfer and Fluid Mechanics Institute,* edited by C. T. Crowe and W. L. Grosshandler, Stanford Univ. Press, Stanford, CA, 1978, pp. 39–54.

Fully Elliptic Incompressible Flow Calculations on Regular Grid

G. V. Hobson*
Naval Postgraduate School,
Monterey, California 93943
and
B. Lakshminarayana†
Pennsylvania State University,
University Park, Pennsylvania 16802

Introduction

ONE of the widely used techniques for the solution of the Navier-Stokes equations is the pressure-based relaxation method, where the flowfield is approximated with an assumed velocity and pressure field and updated using a Poisson solver for the pressure.

Caretto et al.[1] developed an original technique that was a breakthrough at the time, but suffered severely from geometrical limitations because the equations were written in Cartesian coordinates, and the staggered grid arrangement did not allow the easy transformation of the equations to a generalized coordinate system. The pressure correction method (PCM) requires the solution of a Poisson pressure correction equation and a subsequent explicit correction of the velocity and pressure field. Rhie and Chow[2] were the first to compute on a regular, or nonstaggered, grid by substituting the full momentum equations into the integral form of the continuity equation. Their solution procedure, however, still relied on the basic pressure correction algorithm.

Presented as Paper 90-0239 at the AIAA 28th Aerospace Sciences Meeting, Reno, NV, Jan. 8–11, 1990; received May 9, 1991; revision received Feb. 28, 1992; accepted for publication Feb. 28, 1992.

*Associate Professor. Member AIAA.

†Evan Pugh Professor, Department of Aerospace Engineering.

Three different substitution formulations with nonstaggered grids have been suggested by Shih and Ren.[3] Some of these employ the Poisson equation for pressure in place of the continuity equation. Their formulation was derived in nonconservative finite difference form in contrast to the derivation of the equations in conservative form in the present study. The code developed in this paper uses pressure weighting to allow the solution of the discretized equations on a regular grid, and the equations are coupled by the substitution of the pressure-weighted form of the momentum equations into the integral form of the continuity equation. The new method permits the direct solution of the pressure, and not the pressure correction, which allows for the additional coupling of the momentum and pressure equations to be solved in block form.

Theoretical Formulation

The equations governing two-dimensional steady incompressible flow are solved in generalized coordinates since the intended application is for the computation of cascade flows. The discretized form of the momentum equations are evaluated at the control volume faces using pressure weighting. This form of the momentum equation is then substituted into the discretized form of the continuity equation which results in an implicit pressure equation. A rigorous derivation of this method is described by Hobson and Lakshminarayana.[4]

The major difference between this derivation and that of Rhie and Chow[2] is evident when one considers the amount of dissipation in the source term of the implicit pressure equation. The present method does not have as much dissipation for the solution of the pressure equation because it only has two pressure gradient expressions in the source term. The Shih and Ren[3] formulation, which is nonconservative, did not include the pressure-weighted method.

The existence of iterative methods with good error-smoothing properties depends upon the ellipticity of the system of equations. Following Shaw and Sivaloganathan,[5] a linearization of the two-dimensional Navier-Stokes equations in Cartesian coordinates is obtained by freezing ρ, μ at ρ_0, μ_0, respectively, and velocities u,v where they contribute to nonlinear terms, at u_0,v_0, respectively. The linearized system may be written as

$$\begin{bmatrix} c - \mu_0\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] & 0 & \frac{\partial}{\partial x} \\ 0 & c - \mu_0\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

where the linearized convective operator is

$$c = \rho_0\left[u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y}\right] \quad (2)$$

Equation (1) can be written in matrix form as

$$Lq = 0 \quad (3)$$

A Fourier and local mode analysis of this system results in an amplification matrix for this system of equations unique to the discretization scheme. Shaw and Sivaloganathan[5] considered the pressure correction scheme, and Hobson and Lakshminarayana[4] considered the pressure substitution method (PSM).

The distribution of the amplification factor for the PSM and that obtained from the analysis by Shaw and Sivaloganathan[5] for the pressure correction method (PCM)(but for collocated variables) are compared in Table 1. The comparison gives the minimum amplification factors for each method.

Table 1 Amplification factor comparison

	PSM	PCM
$Re_u = Re_v = 1$	0.695	0.591
$Re_u = 100$, $Re_v = 0$	0.754	0.777
$Re_u = Re_v = 100$	0.591	0.624

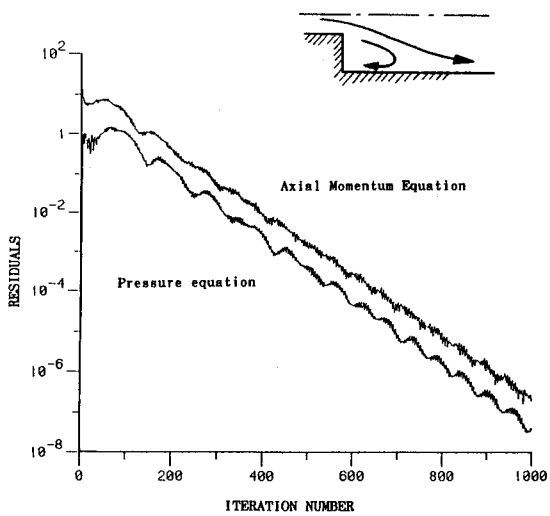

Fig. 1 Convergence history for sudden expansion flow.

Overall the character of the amplification factors for the two methods is similar in shape, which is not surprising since the algorithms only differ in the amount of the dissipation in the pressure equation.

The high frequency spectrum of the PCM is better than the PSM, with a lower amplification factor for the low Reynolds number case. At large cell Reynolds numbers and a purely convective flow, however, the situation is reversed. For this case $Re_u = 100$ and $Re_v = 0$. The PSM has a lower amplification factor in the high frequency region. The difference is not significant, but the trend shows that, for high Reynolds number flows, the PSM is superior to the PCM. This is also evident for the high Reynolds number diffusive case.

Results and Discussion

Although the two-dimensional version of the PSM program has been cast in generalized coordinates, the test case considered is that of flow over a plane symmetric sudden expansion for which flow visualization and laser anemometry measurements are reported by Durst et al.[6] The rectangular computational domain covered half the width of the duct and extended 48 times the upstream channel height H (which is also the step height) downstream of the expansion. Fifteen equally spaced grid points were used in the cross-stream direction and 15 stretched grid points were used in the streamwise direction. The grid stretching factor used in the streamwise direction was 1.4.

This test case was chosen to validate the code by comparing the predicted velocity distributions with those measured at a Reynolds number of 56. The inlet axial velocity distribution used was the same as that measured at a distance of 25% of the step height, upstream of the expansion. Fully developed conditions were assumed to exist at 48 times the upstream channel height H downstream of the expansion. The gradients of all of the flow quantities across the symmetry boundary were set equal to zero along the duct centerline. Relaxation was performed for 1000 outer iterations (five inner iterations were performed for each set of simultaneous linearized equations for u, v, and p), and the convergence of the solution by more than seven orders of magnitude is shown in Fig. 1.

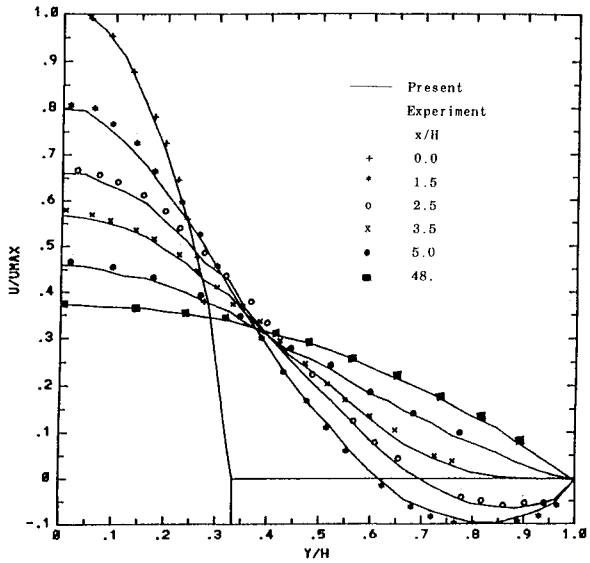

Fig. 2 Comparison between experimental and computational axial velocity profiles for sudden expansion.

The comparison between the predicted and experimentally measured velocity profiles at the various streamwise locations is shown in Fig. 2. The agreement is very good considering the coarse grid used. The present calculations show a slight deficit in the prediction of the centerline velocity. Because the flow velocity in the recirculation region was less than 10% of the centerline velocity, the cell Reynolds numbers in this region were less than 2; thus, false diffusion effects should be negligible.

Conclusion

The present method permits computation on a regular grid instead of on a staggered grid and allows for the solution of the pressure and not the pressure correction. The method closely couples the velocity and pressure field, and thus does not suffer from pressure oscillations.

The local mode analysis showed that the new algorithm has good error smoothing capabilities, especially for high Reynolds number flows. Linear stability analysis indicated that the performance of the PSM and PCM is about the same at low Reynolds numbers. However, at high Reynolds numbers with significant pressure gradients the PSM shows much faster convergence. The overall storage and total CPU time is also reduced with the PSM.

For low Reynolds number laminar flows the PSM showed the same convergence as the PCM; however, for high Reynolds flows the convergence rate of the PSM was better than the convergence rate of the PCM. The new method computed the sudden expansion flow at a low Reynolds number with a very high degree of accuracy.

References

[1]Caretto, L. S., Gossman, A. D., Patankar, S. V., and Spalding, D. B., "Two Calculation Procedures for Steady Three Dimensional Flows with Recirculation," *Proceedings of the Third International Conference on Numerical Methods in Fluid Dynamics*, Springer-Verlag, New York, 1972, pp. 60–68.

[2]Rhie, C. M., and Chow, W. L., "Numerical Study of the Turbulent Flow Past an Airfoil with Trailing Edge Separation," *AIAA Journal*, Vol. 21, 1983, pp. 1525–1532.

[3]Shih, T. M., and Ren, A. L., "Primitive Variable Formulations Using Nonstaggered Grids, *Numerical Heat Transfer*, Vol. 7, 1984, pp. 413–428.

[4]Hobson, G. V., and Lakshminarayana, B., "Fully Elliptic Incompressible Flow Calculations on Regular Grid by a New Pressure Substitution Method," AIAA Paper 90-0239, Reno, NV, Jan. 1990.

[5]Shaw, G. J., and Sivaloganathan, S., "On the Smoothing Proper-

ties of the SIMPLE Pressure-Correction Algorithm," *International Journal for Numerical Methods in Fluids*, Vol. 8, 1988, pp. 441-461.
[6]Durst, F., Melling, A., and Whitelaw, J. H., "Low Reynolds Number Flow over a Plane Symmetric Sudden Expansion," *Journal of Fluid Mechanics*, Vol. 64, 1974, pp. 111-128.

Extension of the λ Formulation to Imperfect Gas Flows

Diego Lentini*
Università degli Studi di Roma "La Sapienza," I-00184 Rome, Italy

I. Introduction

A WIDE class of nozzle flows, including in particular those in air-breathing engines and wind tunnels (except the ones operating in the hypersonic range), involve large temperature excursions, but with stagnation values limited to 2000-2500 K. In such conditions, whereas chemical reactions can still be considered negligible, effects related to the specific heat varying with the temperature cannot. Indeed, specific heat at constant pressure c_p exhibits a fairly large variation with temperature. For example, the c_p of air varies by $\sim 25\%$ in the temperature range 200-2000 K (thermal equilibrium is assumed; see Sec. IV for a discussion of this assumption). At the same time, in the flows under consideration, the thermal equation of state is still closely obeyed. Such behavior must be taken into account in computational models if accurate numerical predictions are required.

The λ formulation[1,2] has proven to be a powerful tool for the numerical solution of compressible flows of a perfect gas. By "perfect" we mean that the gas is assumed to be both thermally and calorically perfect. The first attribute refers to its obeying the perfect gas law, and the latter denotes that its c_p is taken as a constant. This formulation has been successfully extended to finite-rate chemically reacting flows,[3-5] but with the limitation that the component gases of the reacting mixture are perfect.

In this Note we extend the λ formulation to (inert) flows of thermally perfect gas with c_p varying as a function of the temperature T. Such gases are sometimes referred to as "imperfect gases." The resulting formulation is extremely simple and involves a minimum computational overhead in comparison to the perfect gas case. Indeed, existing algorithms for a perfect gas can easily be upgraded with a minimum coding effort, without fundamentally altering their structure.

The proposed formulation is applied to the computation of flow in a quasi-one-dimensional nozzle to prove the workability of this approach and to demonstrate the differences with respect to a perfect gas computation. It will be apparent, however, that the range of application of the present formulation is completely general.

It should be noted that, at temperatures above the range considered here, reaction must inevitably be taken into account. However, this involves completely redefining both the model and the computational algorithm, with associated problems of an entirely different magnitude and much longer computer times. Further, the problem of caloric imperfection must still be addressed at the level of the individual component gases. Instead, the aim of the present paper is to show that within the stated assumptions a significantly better physical description can be achieved at a minimum cost with respect to a standard formulation.

Received May 10, 1991; revision received Feb. 18, 1992; accepted for publication March 7, 1992.

*Researcher, Dipartimento di Meccanica e Aeronautica. Member AIAA.

II. Formulation

The formulation is presented here for simplicity for a quasi-one-dimensional isentropic flow. We assume the speed of sound a as the state variable, and the velocity u as the motion variable. Consequently, the continuity and momentum equations, which we write for convenience in the form

$$a\frac{\rho_t + u\rho_x}{\rho} + au_x = -au\frac{A_x}{A} \tag{1}$$

$$u_t + uu_x + \frac{1}{\rho}p_x = 0 \tag{2}$$

will be recast in terms of the variables a and u. To this end, we observe that the speed of sound is related to the temperature T via the relationship

$$a^2 = \gamma RT \tag{3}$$

where γ is the gas specific heats ratio c_p/c_v, with $R = c_p - c_v$. After logarithmic differentiation we obtain

$$2\frac{a'}{a} = \left(1 + \frac{\gamma_T T}{\gamma}\right)\frac{T'}{T} \tag{4}$$

where the prime denotes differentiation with respect to either t or x, and γ_T is the derivative of γ with respect to T. From the first principle of thermodynamics, written for adiabatic flows with pressure work only,

$$c_v T' = RT\frac{\rho'}{\rho} \tag{5}$$

from which, in view of Eq. (4),

$$\frac{\rho'}{\rho} = \frac{1}{\delta\left(1 + \frac{\gamma_T T}{\gamma}\right)}\frac{a'}{a}$$

with the definition $\delta = (\gamma - 1)/2$. This expression will be substituted into the continuity equation.

The pressure gradient term in the momentum equation can be expressed by means of Gibbs' relationship, which in particular for isentropic flow reduces to

$$\frac{1}{\rho}p' = h'$$

where h is the enthalpy. Then, after Eq. (4), with $h' = c_p T'$,

$$\frac{1}{\rho}p_x = \frac{1}{\delta\left(1 + \frac{\gamma_T T}{\gamma}\right)}aa_x$$

The set of Eqs. (1) and (2) can then be recast as

$$\frac{1}{\delta\left(1 + \frac{\gamma_T T}{\gamma}\right)}(a_t + ua_x) + au_x = -au\frac{A_x}{A} \tag{6}$$

$$u_t + uu_x + \frac{1}{\delta\left(1 + \frac{\gamma_T T}{\gamma}\right)}aa_x = 0 \tag{7}$$

At this juncture, we note that the term a' can be expressed alternatively as $a_T T'$, with a being a function of the temperature T only. We also note that the term

$$\frac{1}{\delta\left(1 + \frac{\gamma_T T}{\gamma}\right)}a_T = \frac{1}{\gamma - 1}\sqrt{\frac{\gamma R}{T}} = \frac{c_p}{a}$$